Solutions' Properties and Numerical Testing of an Interactive Method REF-LEX

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Abstract: *The interactive method REF-LEX is designed to solve nonlinear Multiple Objective Programming Problems (MOPPs). Its basic idea is to use reference directions to search for better solutions according to the Decision Maker's (DM's) preferences and to use lexicographic bi-criterion subproblem for generating efficient solutions. The correctness of the method is presented in this paper on the base of properties of the generated solutions and some numerical test results as well.*
Key words: *Multiple Criteria Programming, Nonlinear Optimization, Interactive Method, Aspiration Level.*

INTRODUCTION

Method REF-LEX was introduced in [2], [3]. In some sense it is inspired from the approach suggested in [6], [12].

The basic characteristics of the method can be summarized as follows. The method includes a dialog in terms of reference points consisting of aspiration levels representing desirable objective values for the DM. The search for the final solution is supported by providing a desired number of compromise solutions along a reference direction by automatically shifting the reference point. This gives the DM a wider selection of solutions and information about the behaviour of the problem and about its possibilities and limitations when compared to other methods based on reference points. Typically, they generate only one solution reflecting the preferences of the DM or shift the reference point artificially. In the REF-LEX method, the DM can specify the reference point in a rather loose way because candidate solutions along the reference direction are generated and one does not have to go as far as the original reference point. The flexibility of the method is important and the DM can decide how many candidate solutions to consider. The solutions are generated using a lexicographic scalarizing problem, which guarantees the efficiency of the solutions. The positive features of the new method include the natural form of dialog with the DM in terms of aspiration levels and the fact that the DM is in control of the method. No artificial parameters or concepts are employed and the DM can direct the search to any part of the efficient set according to his/her preferences.

The base of the method and theoretical comparison with other methods can be found in [2], [3], [4]. and we shall not discussed it here.

The rest of the paper is organized as follow. First, some basic definitions are given. The properties of computed solutions are presented after that and finally the results of computer testing are summarized.

Formulation and Basic Definitions

The learning-oriented method REF-LEX has been introduced for solving nonlinear multiobjective programming problems - MOPPs.

The mathematical formulation of a nonlinear multiobjective optimization problem can be given as:

$$\begin{aligned} &\max\{f_1(x), f_2(x), \dots, f_k(x)\} \\ &\text{subject to} \quad x \in S, \end{aligned} \tag{1}$$

where

a) k (≥ 2) is the number of objective functions,
b) $f_i: R^n \to R$ is a real-valued objective function for $i \in I = \{1, 2, \dots, k\}$,

c) $f(x)=(f_1(x), f_2(x),\dots f_k(x))^T$ is a vector of objective (function) values, or an *objective vector*,
d) $x=(x_1,\dots,x_n)^T$ is an n-dimensional vector of decision variables,
e) $S\subset R^n$ is a feasible set of decision variables formed by constraint functions,
f) R^n and R^k are the decision space and the objective space, respectively and
g) at least one of the objective or constraint functions is nonlinear.

Here the notation "max" means the simultaneous maximization of all the objective functions.

We assume that the objective functions are conflicting, that is, there exists no trivial solution to problem (1) and there is a real need to find the best compromise solution to the problem.

It is often useful to know the best possible values for each objective function. These values form a so-called ideal point z^* in the objective space. Its components are computed as

$$z_i^* = \max_{x\in S} f_i(x), \text{ for all } i \in I.$$

Definition 1. A solution $x\in S$ is *efficient* if and only if there does not exist another solution $y\in S$ such that $f_i(y)\geq f_i(x)$ for all $i\in I$ and $f_j(y)> f_j(x)$ for at least one index j.

The solution is weak efficient if strong inequalities hold for each indices.

Definition 2. A solution $x\in S$ is *properly efficient* if it is efficient and there exists a positive constant M>0 such that for every index i and every solution y for which $f_i(y)>f_i(x)$ there exists at least one index $j\in I$ such that $f_j(y)< f_j(x)$ and $\frac{f_i(y)-f_i(x)}{f_j(x)-f_j(y)} < M$.

Definition 3. A *reference direction* d^h is defined as $d^h = f^r - z^h$ where f^r is a reference point and z^h is a solution chosen at the previous iteration h.

The reference direction is projected in the set of efficient solutions by solving a series of single objective scalarizing problems. These problems are lexicographic. The DM chooses the best solution among the candidates produced and if the most satisfactory solution is found, the solution process stops. Otherwise, the chosen solution is set as the next current solution that will act as the starting point of the new search and the DM needs to set his/her new reference point.

In this learning-oriented method, the DM can experiment with different reference points. In this way, he/she can learn about the problem to be solved and the final solution will actually satisfy him/her. On the other hand, the search along the reference direction can be interpreted as a way of checking the feasibility of the DM's preferences.

The scalarizing problems to be solved are of the form

$$lex\ \min\ \{\max_{i\in I}\ (w_i^{h,l}(z_i^{**} - f_i(x))), \sum_{j=1}^{k} (f_j^{r,l} - f_j(x))\} \qquad \text{(LEX)}$$

$$\text{subject to} \quad x \in S,$$

where $w_i^{h,l}$ are weighting coefficients at iteration h for $i\in I$ and $f_j^{r,l}$ for $j\in I$ are components of different intermediate reference points along the reference direction. Problem (LEX) is solved in two stages. First, the weighted min-max problem is solved subject to the original constraints. This problem is nondifferentiable but it can be formulated as a differentiable problem so that single objective optimizers assuming differentiability can be applied (assuming the original problem is differentiable). In this case, the problem to be solved is of the form

$$\begin{aligned} &\min \alpha \\ &\text{subject to } \; w_i^{h,l}(z_i^{**} - f_i(x)) \le \alpha \;\text{ for all } i \in I \\ &\qquad\qquad x \in S. \end{aligned}$$

This so-called *Tchebycheff problem* has *n*+1 variables. If it has alternative optima, then in the second stage a linear combination of objective functions (i.e., the sum term in (LEX)) is minimized in the set of those solutions. This means that additional constraints are added to the feasible set S. They have the form

$$w_i^{h,l}(z_i^{**} - f_i(x)) \le \overline{\alpha} \;\text{ for all } i \in I,$$

where $\overline{\alpha}$ is the optimal value of α in the first problem. Note that the objective function in the second optimization is used only to break ties in cases of alternative optima.

CORRECTNESS OF THE METHOD

First we show that the scalarizing problem produces Pareto optimal solutions and any Pareto optimal solution can be found.

Theorem 1. The solution of problem (LEX) is Pareto optimal.

Proof: Let $x^* \in S$ be a solution of problem (LEX). Let us assume that it is not Pareto optimal. In this case there exists some other $x' \in S$ such that $f_i(x') \ge f_i(x^*)$ for all $i \in I$ and $f_j(x') > f_j(x^*)$ for at least one index *j*. Because we have $w_i^{h,l} > 0,$ we get $w_i^{h,l}(z_i^{**} - f_i(x')) \le w_i^{h,l}(z_i^{**} - f_i(x^*))$ for all $i \in I$ and, thus, we have $\max_{i \in I} w_i^{h,l}(z_i^{**} - f_i(x')) \le \max_{i \in I} w_i^{h,l}(z_i^{**} - f_i(x^*))$. On the other hand, $\sum_{j=1}^{k}(z_i^{**} - f_i(x')) < \sum_{j=1}^{k}(z_i^{**} - f_i(x^*))$ is valid. Here we have a contradiction with x^* being a solution of (LEX). Thus, x^* is Pareto optimal.

Theorem 2. Let $x^* \in S$ be Pareto optimal. Then there exists a positive weighting vector $w \in R^k$ such that x^* is a unique solution of (LEX).

Proof: Let $x^* \in S$ be Pareto optimal. Let us assume that there exists no positive weighting vector *w* such that x^* is a unique solution of (LEX). We know that $f_i(x) < z_i^{**}$ for all $i \in I$ and for all $x \in S$. Let us set $f_i^r = f_i(x^*)$ for $i \in I$ and $l=N.$ (In what follows, we ignore the index *l*.) Now we have

$$w_i^h = \left[\frac{1}{z_i^{**} - f_i(x^*)}\right] \Big/ \left[\sum_{j=1}^{k} \frac{1}{z_j^{**} - f_j(x^*)}\right]$$

and problem (LEX) can be constructed. Obviously, we can find a value for α such that x^* is a feasible solution of the first part of (LEX) and we can denote the solution pair as (α^*, x^*). We have

$$\alpha^* \ge w_i^h(z_i^{**} - f_i(x^*)) \;\text{ for all } i \in I.$$

After simplifying, we have

$$\alpha^* = \left[\sum_{j=1}^{k} \frac{1}{z_j^{**} - f_j(x^*)} \right]^{-1}.$$

If x^* is not an optimal solution of (LEX), there exists another pair (α', x'), where $x' \in S$, such that this pair is a solution of the lexicographic problem. This means that

$$\alpha^* \geq \alpha' \geq w_i^h (z_i^{**} - f_i(x')) \text{ for all } i \in I.$$

Using the expression for α^* we receive $z_i^{**} - f_i(x^*) \geq z_i^{**} - f_i(x')$ for all $i \in I$. In other words, we have $f_i(x') \geq f_i(x^*)$ for all $i \in I$. Because x^* is Pareto optimal, we must have $f_i(x') = f_i(x^*)$ for all i. In other words, the Tchebycheff problem has a unique optimal solution and, thus, (LEX) must have a unique solution.

An advantage of problem (LEX) is that it does not employ any artificial parameters but still any efficient solution (including properly and improperly ones) can be generated by varying the reference point. This makes the method proposed suitable for solving nonlinear and possibly nonconvex multiobjective optimization problems. In other words, in this method the DM can direct the search in any part of the efficient set according to his/her preferences.

NUMERICAL TESTING AND AN ILLUSTRATIVE EXAMPLE

Test problems were mostly academical. Some of them are published in [1], [7]-[11].

The dimension of problems vary from 2 to 5 nonlinear objective functions, the feasible set was defined from 1 to 6 constraints (nonlinear and linear) and decision variables no more than 5.

The experiments show that the method did work fine with test examples. Usually it was easy for the decision maker to find satisfying compromise solution. Only sometimes those functions whose values were allowed to worsen did worsen more than expected.

The idea of using weight coefficients 1/L for indexes j such that aspiration level equals utopian (modified ideal point) value seemed to work fine too. This situation only occurred in tests when DM gave aspiration levels equal to utopian values.

It seemed that if DM gave aspiration levels close to utopian value for some functions the method seemed to improve almost only the values of those functions while other function values could worsen a lot.

In evaluation step it would be helpful if there existed one more option for DM to choose: going back to previous computed solution. The values of those functions which DM allows to increase do sometimes worsen more than expected. (Of course the same can be done by giving a new direction with the function values of previous solution as aspiration levels.)

In all test examples the objective functions are to be MINIMIZED.

Next we illustrate the REF-LEX method by solving a water quality management problem involving seven objective functions, three variables, one nonlinear constraint and box constraint. For details of the problem, we refer to Miettinen and Mäkelä, 1999 where the problem has been solved using three different methods.

The problem describes the pollution problems of an artificial river basin. A cannery and two cities pollute the river and a park is located between the cities. The water quality is measured by dissolved oxygen (DO) concentration. The cannery and the cities already reduce the waste content but new treatment facilities are needed. However, their costs will reduce the investment return from the cannery and increase the tax rate in the cities. The three decision variables represent the treatment levels of waste discharge at the cannery and in the cities, and the nonlinear constraint restricts the DO level at the end of the river.

The first and the third objective functions describe the DO concentration in the cities and the second objective in the park whereas the fourth objective represents the percent return on investment at the cannery. The addition to the tax rate in the cities is modeled as the fifth and the sixth objective, respectively, and the last objective keeps the capacities of the new treatment facilities as close to their optimal level as possible. Note that the last three objective functions were originally to be minimized and their signs have here been changed.

In Miettinen and Mäkelä, 1999, the general preferences of the DM are described as follows. The DO level in the cities and in the park should be at least 6.0 mg/l, the rate of return on investment at the cannery should be above 6.5%, the tax rates in the cities should be below $1.5/$1000 of assessed valuation, and the capacities of the new treatment facilities should be at most 20% from the optimal level. Note that the goals are not necessarily attainable simultaneously. The ideal point is here (6.34,6.79,6.60,7.50,0.00, -0.96,-0.16). Next we solve the problem following the preferences and choices expressed in the reference.

Following the reference, we start the solution process from the feasible point (0.41,0.45,1.00) and the corresponding objective vector z^1 =(5.00,2.55,5.29,7.44,-0.12,-11.37,-0.35). Here, for example, the tax rate in the second city is far too high and the DO concentration in the park is unsatisfactory. We use the aspiration levels specified as a basis for the first reference point. However, because the value of the fourth objective function is so good, we do not want to relax it too much and set the corresponding aspiration level as -7.0. Thus, the first reference point is f^r = (6.00,6.00,6.00,7.00,-1.50,-1.50,-0.20). We ask for five candidate solutions (i.e. *N*=5) in order to learn about the behavior of the problem.

The first four candidate solutions are $z^{1,1}$ = (5.23,3.26,5.52,7.35,-0.39,-5.87,-0.31), $z^{1,2}$ = (5.41,3.77, 5.69,7.25,-0.71,-4.00,-0.28), $z^{1,3}$ = (5.58,4.23,5.83.7.12,-1.12,-2.72,-0.24) and $z^{1,4}$ = (5.74,4.70,5.98, 6.92,-1.73,-2.47,-0.23). It does not seem interesting to calculate the last candidate because the values of the other objectives are rather good but the value of the sixth objective does not seem to decrease enough. Thus, $z^{1,4}$ is selected as the current solution z^2 and a new reference point f^r = (6.00,5.00,6.00,6.50,-2.00, -1.70,-0.22) is set. Because the aspiration levels are close to the desired values, we generate only one candidate solution $z^{2,1}$ = (5.97,4.83,6.02,6.40,-1.69,-1.70,-0.20). We set this solution as z^3 and even though it is already relatively good, the DO concentration in the park should be better. A new reference point f^r = (6.00,5.00,6.00,6.30,-2.00,-1.50,-0.22) is specified. When one candidate solution is generated, we get $z^{3,1}$ = (6.00,4.97,6.06,6.28,-1.93,-1.50,-0.22), which is selected as the final solution. The corresponding decision variable values are (0.85,0.87,0.81).

When comparing the solution obtained to those mentioned in Miettinen and Mäkelä, 1999, we can say that we found a very good solution. The best solution found in the reference is (6.01,5.00,6.07,6.26,-1.98, -1.54,-0.22). In our solution with REF-LEX, we managed to improve tax rates in both the cities without sacrificing too much in the other objectives. As far as computational cost is concerned, only 233 function evaluations were needed. Besides, REF-LEX obeyed the preferences expressed in the form of reference points very well and the DM could feel that his/her hopes were reflected in the solutions obtained.

CONCLUSION

In this paper we show that the method REF-LEX generates only efficient solutions and that every efficient solution can be generated by it at some iteration. In other words, the whole efficient set can be successively computed.

Also, as the numerical experiments show, the method seems to be flexible and well represents the DM's preferences. It can be used for solving nonconvex nonlinear MOPPs due to used lexicographic subproblem.

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