

Approximation Algorithms for Steady-State Solutions of Markov Chains

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Abstract: Two main approximation methods for steady-state analysis of Markov chains are introduced: Courtois's decomposition method and Takahashi's iterative method. The Courtois's approach is based on decomposability properties of the models under consideration. While Courtois's method is non-iterative and is applied for the approximate computation of the steady-state probability vector for a given ergodic Discrete-Time Markov Chain -DTMC (or Continuous-Time Markov Chain), Takahashi's iterative method allows computation of the exact state probability vector. Examples and numerical results are shown.

Key words: Queuing Theory, Markov Chains, Numerical Methods, Approximation Techniques

INTRODUCTION

The two basic approximation methods for steady-state analysis of Markov chains [1] can be used for analysis of communication systems, presented as Markov chains. The method of Courtois is mainly applied to approximate computations \mathbf{v}^{\approx} of the desired state probability vector \mathbf{v} . The efficiency of Courtois's method is due to the fact that instead of solving one linear system of equations with size of the state space S , several much smaller linear systems are solved independently. One system is solved for each subset S_j of the partitioned state space S , and one for the aggregated chain [2,7].

Takahashi's method differs substantially from Courtois' approach both with respect to the used methodology and the applicability conditions. While Courtois's method is non-iterative and is applied for the approximate computation of the steady-state probability vector \mathbf{v}^{\approx} for a given ergodic DTMC (or $\boldsymbol{\pi}^{\approx}$ in the case of a CTMC), Takahashi's iterative method allows a computation of the exact state probability vector [3]. Numerical errors are still encountered even in such "exact" methods. The method is exact in that there are no modelling approximations. To allow a straightforward comparison of the two methods, any given ergodic CTMC can easily be transformed into an ergodic DTMC [4].

COURTOIS'S APPROXIMATION METHOD

Courtois's approach is based on decomposability properties of the models under consideration. Initially substructures that can separately be analyzed are identified. Then, an aggregation procedure is performed that uses independently computed sub results as constituent parts for composing the final results. The applicability of the method needs to be verified in each case. If the Markov chain has tightly coupled subsets of states, where the states within each subset are tightly coupled to each other and weakly coupled to states outside the subset, it provides a strong intuitive indication of the applicability of the approach. Such a subset of states might then be aggregated to form a macro state as a basis for further analysis. The macro state probabilities, together with the conditional micro state probabilities from within the subsets, can be composed to yield the micro state probabilities of the initial model. Let \mathbf{P} be a transition probability matrix. The state space associated with \mathbf{P} is partitioned into groups, often referred to as lumps, such that any state within a group contains the same number of customers in a specific portion of the state descriptor [8]. The transition probability matrix \mathbf{P} , written out according to these groups consists of blocks of transition probabilities along the diagonal and of other blocks along the off-diagonals above and below the diagonal. \mathbf{P} is said to be *Nearly Completely Decomposable* (NCD) if the sum of the non-zero transition probabilities on each row that lie within the diagonal block is close to 1. That means that the sum of the off-diagonal probabilities along a row is extremely small. This occurs when most of the transitions are between states of the same group, with very few transitions between states of different groups [6]. The error induced by the method can, in principle, be bounded, but a formal error bounding is often omitted. A modified algorithm for calculation of probability vector, built according to Courtois's approximation method is presented in Figure 1.

Step 1 Create the state space and arrange it suitably according to a pattern of decomposition.

Step 2 Build the transition probability matrix \mathbf{P} (use a randomization technique $\mathbf{P}=\mathbf{Q}/q+\mathbf{I}$ if the starting point is a CTMC), and partition \mathbf{P} into $M \times M$ number of submatrices \mathbf{P}_{IJ}^* , $0 \leq I, J \leq M-1$.

Step 3 Verify the nearly complete decomposability of \mathbf{P} according to

$$\varepsilon < \frac{1 - \max_i |\lambda_i^*(2)|}{2} \text{ with the chosen value of } \varepsilon.$$

Step 4 Decompose \mathbf{P} such that $\mathbf{P} = \mathbf{P}^* + \varepsilon \cdot \tilde{\mathbf{C}}$. Matrix \mathbf{P}^* contains only stochastic diagonal submatrices \mathbf{P}_{II}^* , and ε is a measure of the accuracy of Courtois's method. It is defined as the maximum sum of the entries of the non-diagonal sub-matrices \mathbf{P}_{IJ}^* , $I \neq J$ of \mathbf{P} .

Step 5 For each I , $0 \leq I, J \leq M-1$, solve equation $\mathbf{v}_I^* \cdot \mathbf{P}_{II}^* = \mathbf{v}_I^*$, with $\mathbf{v}_I^* \cdot \mathbf{1} = 1$ to obtain the conditional state probability vectors \mathbf{v}_I^* .

Step 6 Compute the coupling between the decomposed macro states:

Step 6.1 Generate the transition probability matrix $\mathbf{\Gamma} = [\Gamma_{IJ}]$ according to

$$\text{equation } \Gamma_{IJ} = \sum_{i \in S_I} \left(\mathbf{v}_{Ii}^* \sum_{j \in S_J} p_{Ijij} \right).$$

Step 6.2 Solve $\boldsymbol{\gamma} \cdot \mathbf{\Gamma} = \boldsymbol{\gamma}$, to obtain the macro steady-state probability vector $\boldsymbol{\gamma}$.

Step 7 Compute the approximate steady-state probability vector \mathbf{v}^{\approx} of the micro states by unconditioning the conditional state probability vectors \mathbf{v}_I^* , $0 \leq I \leq M-1$, according to $\mathbf{v}_{Ii}^{\approx} = \boldsymbol{\gamma}_I \mathbf{v}_{Ii}^*$, $0 \leq I \leq M-1$ and $\forall i \in S_I$.

Step 8 From \mathbf{v}^{\approx} compute steady-state performance and dependability measures along the lines of a specified reward structure.

Figure 1. Algorithm 1 for approximate calculating of probability vector

TAKAHASHI'S ITERATIVE METOD

Compared to Courtois's method, Takahashi's approach partitions the state space S into M disjoint subsets of states $S_I \subset S$, $0 \leq I \leq M-1$ such that each subset S_I is aggregated into a macro state I . The criteria, however, used to cluster states differ in the two approaches. The calculation of transition probabilities Γ_{IJ} among the macro states I and J is performed on the basis of conditional micro-state probability vector \mathbf{v}_I^* and the originally given transition probabilities p_{ij} , or p_{Ijij} [5].

With Courtois's method, the conditional probability vectors \mathbf{v}_I^* , for given partition element I , can be separately computed for each subset of micro states $S_I \subset S$, $0 \leq I \leq M-1$, if the original model is nearly completely decomposable.

By contrast, in Takahashi's method, the partitioning of the state space is performed on the basis of approximate lumpability. Takahashi's method was presented in terms of DTMCs. However this choice implies no limitation since it is a well-known fact that ergodic DTMCs and CTMCs are equivalent with respect to their steady-state probability computations, when applying transformations.

But Takahashi's method can be more conveniently applied directly on the generator matrix $\mathbf{Q} = [q_{ij}]$ of a CTMC. Takahashi's iterative method allows a computation of the exact state probability vector. In Takahashi's method the partitioning of the state space is performed on the basis of approximate lumpability. A modified algorithm, built according to Takahashi's iterative method is presented in Figure 2.

Step 1 Create the state space S and arrange it suitably according to a pattern of decomposition along the lines of approximate lumpability into transition probability matrix.

Step 2 Initialization:

2.1 $n := 0$

2.2 estimate \mathbf{v}^0

2.3 choose ε and $0 < c < 1$

2.4 choose some vector norm function $f(\|\dots\|)$;

Step 3 $f(\|\mathbf{v}^{(n)} - \mathbf{v}^{(n)}\mathbf{P}\|) \geq \varepsilon$

Step 4 Geometric convergence:

from Y to N : $n \geq 1$

from Y to N : residual error $(\mathbf{v}^{(n)}) > c \cdot \text{residual error}(\mathbf{v}^{(n-1)})$

$\mathbf{v}' = \mathbf{v}^{(n)}$; $\mathbf{v}^{(n)} = \mathbf{v}'\mathbf{P}$; $n := n+1$;

Step 5 Aggregation for all: $0 \leq i, j \leq M-1$

5.1 Create $\Gamma^{(n)} = [\Gamma_{ij}^{(n)}] = \left[\sum_{i \in S_i} \sum_{j \in S_j} \frac{\mathbf{v}_i^{(n-1)} p_{ij}}{\sum_{k \in S_i} \mathbf{v}_k^{(n-1)}} \right]$;

5.2 Solve $\boldsymbol{\gamma}^{(n)} = \boldsymbol{\gamma}^{(n)} \cdot \Gamma^{(n)}$ according to $\boldsymbol{\gamma} = \boldsymbol{\gamma} \cdot \Gamma$, $\boldsymbol{\gamma} \cdot \mathbf{1} = 1$;

Step 6 Disaggregating for all: $0 \leq i \leq M-1$

6.1 Calculate $\Gamma_{ij}^{(n)} = \sum_{i \in S_i} \frac{\mathbf{v}_i^{(n-1)} p_{ij}}{\sum_{k \in S_i} \mathbf{v}_k^{(n-1)}}$, $\forall j \in S$ according to $\Gamma_{ij} = \sum_{i \in S_i} \frac{\mathbf{v}_i p_{ij}}{\sum_{k \in S_i} \mathbf{v}_k}$.

6.2 Calculate $\mathbf{v}_i^{(n)} = [\mathbf{v}_{ij}]$ by solving the system of equations:

$$\mathbf{v}_i^{(n)} = \sum_{j \in S_j} \mathbf{v}_j^{(n)} p_{ji} + \sum_{K=0, K \neq i}^{M-1} \boldsymbol{\gamma}_K^{(n)} \Gamma_K^{(n)} \mathbf{v}_i, \quad \forall i \in S_i;$$

Step 7 With $\mathbf{v}^{(n)} = [\mathbf{v}_i^{(n)}]$ from the previous step solve: $\mathbf{v}^{(n)} \cdot \mathbf{1} = 1$.

Figure 2 Algorithm 2 for iterative calculation of probability vector

NUMERICAL EXAMPLE FOR ALGORITHMS IMPLEMENTATION

Let assume a system in which two customers are circulating among three stations according to some stochastic regularities, as shown on Figure 3. In the interesting case of a network of queues we define a local-balance equation (with respect to a given network state and a network node i) as one that equates the rate of flow out of that network state due to the departure of a customer from node i to the rate of flow into that network state due to the arrival of a customer to node i .

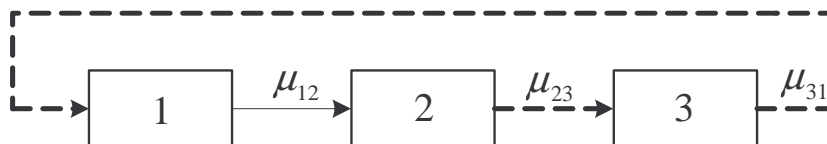


Figure 3 A Jackson cyclic network with three stations and two users

Each arbitrary pattern of distribution of the customers among the stations is represented by a state. In our example we have 6 states, as shown on Figure 4. In state 2,0,0 for example, two customers are in station one, while stations two and three are both empty. After an exponentially distributed time period, a customer travels to state two. The transition behavior is governed by the transition rate μ_{12} . The transition behaviour between the other states can be explained similarly. The arrows point the direction for the possible transitions, as only counter clockwise direction is allowed in the network. Further, we

assume that the customers preferably stay in two stations and relatively rarely transfer to the third station, which is the most isolated one.

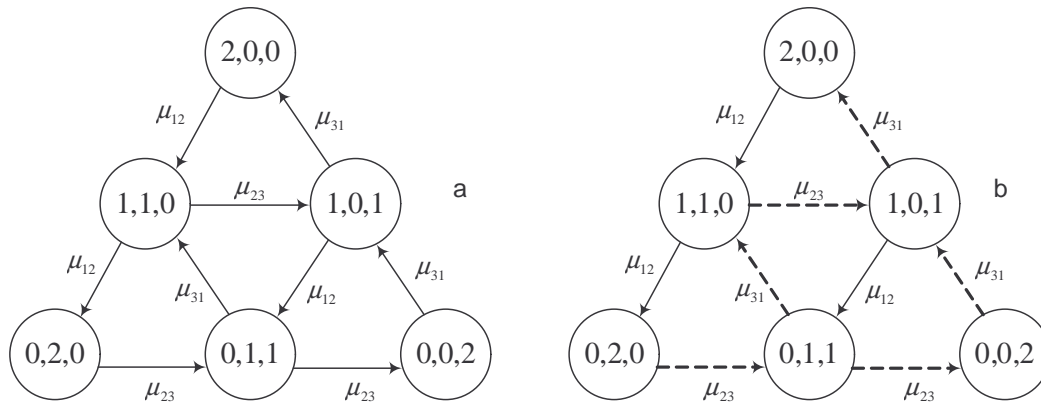


Figure 4. a) State transition rate diagram for tightly coupled states and b) loosely coupled states

Solid arcs emphasize the tightly coupled states, while the dotted ones represent the loose coupling. The parameters with strong interaction between state 1 and 2, while state 3 interacts less with the others and their transition rates are for $\mu_{12} = 4$, $\mu_{21} = 0$, $\mu_{13} = 0$, $\mu_{31} = 0.2$, $\mu_{23} = 0.4$, $\mu_{32} = 0$. The corresponding infinitesimal generator matrix \mathbf{Q} is shown in (1).

$$\mathbf{Q} = \begin{pmatrix} -4 & 4 & 0 & 0 & 0 & 0 \\ 0 & -4.4 & 4 & 0.4 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0.4 & 0 \\ 0.2 & 0 & 0 & -4.2 & 4 & 0 \\ 0 & 0.2 & 0 & 0 & -0.6 & 0.4 \\ 0 & 0 & 0 & 0.2 & 0 & -0.2 \end{pmatrix} \quad (1)$$

The values for generator matrix \mathbf{Q} are depicted symbolically in Table 1:

Table 1: Generator matrix \mathbf{Q} of the CTMC structured for decomposition

State	(2,0,0)	(1,1,0)	(0,1,0)	(1,0,1)	(0,1,1)	(0,0,2)
(2,0,0)	$-\sum$	μ_{12}	0	μ_{13}	0	0
(1,1,0)	μ_{21}	$-\sum$	μ_{12}	μ_{23}	μ_{13}	0
(0,2,0)	0	μ_{21}	$-\sum$	0	μ_{23}	0
(1,0,1)	μ_{31}	μ_{32}	0	$-\sum$	μ_{12}	μ_{13}
(0,1,1)	0	μ_{31}	μ_{32}	μ_{21}	$-\sum$	μ_{23}
(0,0,2)	0	0	0	μ_{31}	μ_{32}	$-\sum$

Given the condition $q > \max_{i,j \in S} |q_{ij}| = -4.4$ we fix $q = 5$, which leads to the transition probability matrix \mathbf{P} , presented in (2).

$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.12 & 0.8 & 0.08 & 0 & 0 \\ 0 & 0 & 0.92 & 0 & 0.08 & 0 \\ 0.04 & 0 & 0 & 0.16 & 0.8 & 0 \\ 0 & 0.04 & 0 & 0 & 0.88 & 0.08 \\ 0 & 0 & 0 & 0.04 & 0 & 0.96 \end{pmatrix} \quad (2)$$

To derive stochastic submatrices, the transition probability matrix \mathbf{P} is decomposed into two matrices \mathbf{A} and \mathbf{B} : $\mathbf{P}=\mathbf{A}+\mathbf{B}$, according to (3).

$$\mathbf{P} = \underbrace{\begin{pmatrix} 0,2 & 0,8 & 0 & & & & \\ 0 & 0.12 & 0.8 & & 0 & & \\ 0 & 0 & 0.92 & & & & \\ & & & 0.16 & 0.8 & & \\ & & & 0 & 0.88 & & \\ & & & & & 0.96 & \end{pmatrix}}_{\mathbf{A}} + \underbrace{\begin{pmatrix} & & & & 0 & 0 & 0 \\ & & & & 0 & 0.08 & 0 & 0 \\ & & & & & 0 & 0.08 & 0 \\ 0.04 & 0 & 0 & 0 & & & & 0 \\ 0 & 0.04 & 0 & & & & & 0.08 \\ 0 & 0 & 0 & 0.04 & 0 & & & 0 \end{pmatrix}}_{\mathbf{B}} \quad (3)$$

Next we define a matrix \mathbf{X} , such that $\mathbf{P} = (\mathbf{A} + \mathbf{X}) + (\mathbf{B} - \mathbf{X}) = \mathbf{P}^* + \mathbf{C}$, according to (4).

$$\mathbf{P} = \underbrace{\begin{pmatrix} 0,2 & 0,8 & 0 & & & & \\ 0.08 & 0.12 & 0.8 & & 0 & & \\ 0.02 & 0.06 & 0.92 & & & & \\ & & & 0.16 & 0.84 & & \\ & & & 0 & 0.12 & 0.88 & \\ & & & & & & 1 \end{pmatrix}}_{\mathbf{P}^*} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.08 & 0 & 0 & 0.08 & 0 & 0 & \\ 0.02 & 0.06 & 0 & 0 & 0.08 & 0 & \\ 0.04 & 0 & 0 & 0 & 0.04 & 0 & \\ 0 & 0.04 & 0 & 0.12 & 0 & 0.08 & \\ 0 & 0 & 0 & 0.04 & 0 & 0.04 & \end{pmatrix}}_{\mathbf{C}} \quad (4)$$

As a first step toward the computation of the approximate state probability vector $\mathbf{v}^* \approx \mathbf{v}$, each submatrix \mathbf{P}_{ii}^* is analyzed separately and the conditional state probability vector \mathbf{v}_i^* , $0 \leq i \leq M-1$ is computed, as shown in (5).

$$(\mathbf{v}_{00}^*, \mathbf{v}_{01}^*, \mathbf{v}_{02}^*) \cdot \begin{vmatrix} (0.2-1) & 0.8 & 0 \\ 0.08 & (0.12-1) & 0.8 \\ 0.02 & 0.06 & (0.92-1) \end{vmatrix} = 0, \quad \mathbf{v}_0^* \cdot \mathbf{1} = 1 \quad (5)$$

This yields the conditional steady-state probabilities of the micro states aggregated to the corresponding macro state 0, as shown in (6).

$$\mathbf{v}_{00}^* = 0.0308, \quad \mathbf{v}_{01}^* = 0.0881, \quad \mathbf{v}_{02}^* = 0.8811. \quad (6)$$

Similarly are received values (7).

$$\mathbf{v}_{10}^* = 0.125, \quad \mathbf{v}_{11}^* = 0.875 \quad \text{and} \quad \mathbf{v}_2^* = \mathbf{v}_{20}^* = 1. \quad (7)$$

Next, are derived the macro-state transition probabilities and the macro-state transition probability matrix $\mathbf{\Gamma}$, as follows from (8),

$$\begin{matrix} \Gamma_{00} = 0.9225 & \Gamma_{10} = 0.04 & \Gamma_{20} = 0 \\ \Gamma_{01} = 0.0775 & \Gamma_{11} = 0.89 & \Gamma_{21} = 0.04 \\ \Gamma_{02} = 0 & \Gamma_{12} = 0.07 & \Gamma_{22} = 0.96 \end{matrix} \quad \mathbf{\Gamma} = \begin{pmatrix} 0.9225 & 0.0775 & 0 \\ 0.04 & 0.89 & 0.07 \\ 0 & 0.04 & 0.96 \end{pmatrix} \quad (8)$$

from which the macro steady state probabilities are calculated as (9).

$$\gamma_0 = 0.158, \quad \gamma_1 = 0.306, \quad \gamma_2 = 0.536. \quad (9)$$

The final step is to obtain the approximate steady-state probabilities for the original model, shown in Table 1. In order to make a comparison, we calculate the state probabilities, using a direct method. The results show that the approximated probabilities

achieved by Courtois's method resemble the exact ones to the second decimal digit, as is seen from Table 2, where are given received results according to three approaches.

Table 2. Comparison of numerical results for calculated steady-state probabilities

State	Exact	Cortois		Takhashi			
		\mathbf{v}^{\approx}	Error	$\mathbf{v}^{(0)}$	$\mathbf{v}^{(1)}$	$\mathbf{v}^{(2)}$	Error
(2,0,0)	0.0014	0.0049	2.56	0.166	0,0112	0,0015	-0,0714
(1,1,0)	0.0137	0.0139	0,017	0.166	0,0203	0,0137	0
(0,2,0)	0.1368	0.1391	0,017	0.166	0,2030	0,1378	-0,0073
(1,0,1)	0.0274	0.0382	0.398	0.166	0,0319	0,0273	0,0036
(0,1,1)	0.2736	0.2679	-0.02	0.166	0,2871	0,2739	-0,0011
(0,0,2)	0.5471	0.5358	-0.02	0.166	0,4466	0,5457	0,0026

Better results are achieved when using more levels of decomposition. The state probabilities vary significantly from each other, which is another obstacle for achieving best results. It is obvious that Takahashi's method shows good results. After the first iteration step, the results are not near to the exact values, but after the second iteration they resemble the exact values up to the third decimal digit.

CONCLUSIONS

Two main approximation methods for steady-state analysis of Markov chains are discussed. The method of Courtois is applied for the approximate computation of the steady-state probability vector, while Takahashi's iterative method allows a computation of the exact state probability vector. These methods are quite useful for quick and easy solution of models with large state spaces, if the underlying model is nearly completely decomposable. If the probability matrix obeys certain regularity conditions the results can be exact. For better results the iteration steps can be repeated. A system with two customers, circulating among three stations according to some stochastic regularity is reviewed. Built algorithms for steady-state analysis according to Courtois and Takahashi's methods allow calculating approximately the probability vector of Markov chains.

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