Fourier and Hilbert Transforms

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Abstract: *In the paper two types of discrete transforms - Fourier transform and Hilbert transform are analysed. These transforms are useful for many applications. It is shown that the analysing filters of the Discrete Fourier Transform (DFT) can be constructed applying the Hilbert transform. This approach gives us a new representation of the full recursive form of the Fast Fourier Transform (FFT). The results give new possibilities for fast realizations of the DFT.*

Key words: Discrete transforms, Fast Fourier transform (FFT), Hilbert transform, signal processing, fast transforms, filter banks, characters of groups and theory of groups.

INTRODUCTION

Signal processing has entered a period of comprehending its different parts in more coherent structure. In our age of fast progress in computer sciences "what brings these parts together and integrates them is hard to be overestimated". Abstract harmonic analysis, which underlies linear signal processing technology, provides us with the tools we need. We are not only interested in the computational aspects of the algorithms, but also in their algebraical, geometrical and physical representations, which is the basis for further development. We present in this paper relations between the Fourier Transform and Hilbert Transform in connection with the fast realizations of these transforms.

CIRCULAR CONVOLUTION

The input and output signals of a linear time-invariant system are connected by the convolution operation [1]:

$$\mathbf{y} = \mathbf{x} * \mathbf{h}. \tag{1}$$

Here **h** is the impulse response of the system. The sets of the real numbers **R,** integer numbers **Z** and the integer numbers – multiple of some integer number **n** (i.e. **nZ),** with addition as a binary operation, are groups [2]. The signals are functions usually defined on the **R**, **Z**, the torus group **T** = **R**/**Z**, the residue system (mod n) **Z**/n**Z**, or their Cartesian products. As such they are elements of some functional space, most often a Hilbert space **H**, which is supplied with the form **(x|y)** that takes values in the field of the complex numbers **C**. This form, called an inner or scalar product, is Hermitian and positive definite [4].

If $\mathbf{L}^2(\mathbf{Z}/n\mathbf{Z})$ denotes n-dimensional complex vector space of functions (vectors), than we have a Hilbert spaces with an inner product [1][3][4]

$$(\vec{\mathbf{x}} | \vec{\mathbf{y}}) = \sum_{\mathbf{k} \in \mathbf{Z}/n\mathbf{Z}} \mathbf{x}_\mathbf{k}^* . \mathbf{y}_\mathbf{k} \ . \tag{2}$$

The circular convolution (1) could be written as an inner product, if the right shift operator ρ and the sign operator σ are used. In the canonical basis of $\mathbf{L}^2(\mathbf{Z}/n\mathbf{Z})$, formed by vectors, $\{\vec{\mathbf{e}}_\mathbf{k}\}$, $\vec{\mathbf{e}}_\mathbf{k} = \left[\delta_{\mathbf{l},\mathbf{k}}\right]$, l, k = 0,1,...,(n-1) (mod n), ($\delta_{l,k}$ is the Kronecker's symbol), the two endomorphisms have the following (orthogonal) matrices [3]:

$$\rho_n = \lfloor \delta_{k-1,l} \rfloor,\ \sigma_n = \lfloor \delta_{k,n-l} \rfloor,\quad k,l = 0,1,\ldots,n-1 \pmod n$$

$$\rho_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad \sigma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{3}$$

In this case the k-th coordinate of the output vector $\vec{\mathbf{y}}$ in (1) has the form

$$y_k = (\vec{\mathbf{x}} \,|\, \rho^k \sigma \vec{\mathbf{h}}), \tag{4}$$

and the convolution is cyclic. In (4) ρ and σ define a linear representation of the dihedral group $\mathbf{D}_n$ [2][3][7][8]:

$$D_n = \langle \sigma, \rho \,|\, \sigma^2 = \rho^n = (\sigma\rho)^2 = 1 \rangle. \tag{5}$$

DISCRETE FOURIER TRANSFORM

The discrete Fourier operator $\mathbf{F}_n$ is defined by the square matrix

$$\mathbf{F}_n = \frac{1}{\sqrt{n}} \sum_{0 \le k,l < n} e^{-j\frac{2\pi k l}{n}} \rho^k \vec{\boldsymbol{\delta}}\, \vec{\boldsymbol{\delta}}^T \rho^{-l},$$

$\mathbf{F}_n = \frac{1}{\sqrt{n}}\left[w^{kl}\right] = C_n - jS_n$, $w = e^{-j2\pi/n}; k,l = 0,1,\ldots,n-1$; this operator is unitary [1][4], i.e. the inverse one coincides with the Hermitian conjugate: $\mathbf{F}^{-1} = \mathbf{F}^*$. In the abstract harmonic analysis discrete transform is considered to be an expansion by characters of the cyclic group **Z**/n**Z** [2][7][8]. A fundamental property of the Fourier operator is that it transforms convolution into algebraic multiplication [1], i.e. **F**(x∗h) = **F**x.**F**h.

Let us introduce (by n = 2^m) the following rectangular ($\frac{n}{2} \times n$) "selection" matrices:

$$\mathbf{E}_{n/2} = \left[\delta_{2i,k}\right];\ \mathbf{O}_{n/2} = \left[\delta_{(2i+1),k}\right];\ \mathbf{U}_{n/2} = \left[\delta_{i,k}\right];\ \mathbf{L}_{n/2} = \left[\delta_{(i+n/2),k}\right];\ i = 0,1,\ldots,\frac{n}{2}-1;\ k = 0,1,\ldots,n-1. \tag{6}$$

For n = 4 we have

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix};\ \mathbf{O} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix};\ \mathbf{L} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The so introduced matrices have the following important properties:

$$\mathbf{E^T E + O^T O = 1;\ U^T U + L^T L = 1;\ E\,E^T = O\,O^T = U\,U^T = L\,L^T = 1;\ E\,O^T = L\,U^T = 0.}$$

The full recursive form of the Fast Fourier Transform (FFT) is found in [3]:

$$\sqrt{2}\ \mathbf{F_n} = \mathbf{(U+L)^T F_{n/2} E + (U-L)^T} \mu_{n/2}^{-1/2} \mathbf{F_{n/2} O}, \tag{7}$$

$$\mu_n = \mathrm{diag}(1, e^{j\frac{2\pi}{n}}, \ldots, e^{j\frac{2\pi}{n}k}, \ldots, e^{j\frac{2\pi}{n}(n-1)}).$$

DISCRETE HILBERT TRANSFORM

The Hilbert transform (or more correctly endomorphism) **κ** (kappa) is applied in many areas: generating of single-sideband signals, inverse filtering, image processing, speech processing, radiolocation, compressing and etc. [5].
Let the dimension of the signals' (vectors') space n be an even number, and the discrete delta (vector) of Dirac has the form:

$$\vec{\boldsymbol{\delta}} = [\,1,\ 0,\ 0, ...,\ 0]^T .$$

In that case the sign vector by definition has the form:

$$\vec{\mathbf{s}} = \frac{1}{\sqrt{n}}(1-\rho^{n/2}) \sum_{0<\ k<n/2} \rho^k \vec{\boldsymbol{\delta}} \qquad (8)$$

For n = 8 this vector looks like this:

$$\vec{\mathbf{s}} = \frac{1}{\sqrt{8}}[0,\quad 1,\quad 1,\quad 1,\quad 0,\quad -1,\quad -1,\quad -1]^T .$$

If n is odd, the "middle" zero will disappear. This vector is odd, i.e. $\sigma\, \vec{\mathbf{s}} = -\vec{\mathbf{s}}$.
One can obtain:

$$\vec{\kappa} = \mathbf{F}_n^{\ *}(-j\ \vec{\mathbf{s}}) = \frac{2}{n} \sum_{0\le k<n/2} \text{ctg}(\frac{\pi}{n}(2k-1))\, \rho^{2k-1}\ \vec{\boldsymbol{\delta}}\ ,$$

$$\kappa(\rho) = \frac{2}{n} \sum_{0\le k<n/2} \text{ctg}(\frac{\pi}{n}(2\,k-1))\, \rho^{2\,k-1}\ . \qquad (9)$$

$$\kappa^2(\rho) = -1 + \frac{2}{n} \sum_{0\le k<n/2} \rho^{2\,k} = -1 + \frac{1}{n}\vec{\mathbf{1}}\vec{\mathbf{1}}^T + \mu^{n/2}\frac{1}{n}\vec{\mathbf{1}}\vec{\mathbf{1}}^T \mu^{-n/2}$$

The first row of (9) is the impulse response, and the second one is the cyclic discrete endomorphism (system function) of Hilbert (an ideal cyclic Hilbert transformer or 90 degree phase shifter), that is antisymmetric and (anti-) commute with σ, i.e.

$$\kappa^T = -\,\kappa = \sigma\,\kappa\,\sigma\,, \quad \Rightarrow \sigma\,\kappa = -\,\kappa\,\sigma, \quad \Rightarrow \sigma\,\kappa^2 = \kappa^2\,\sigma.$$

In (9) $\vec{\mathbf{1}}$ is the vector of all 1's.

THE ANALYSING FILTERS OF THE FOURIER TRANSFORM

The analysing filters of the Fourier Transform are found in [6]. Let us write (7) in the following form:

$$\sqrt{2}\,\mathbf{F_n} = \mathbf{U^T\, F_{n/2}\, [E} + \rho^{\mathbf{1/2}}\mathbf{O]} + \mathbf{L^T F_{n/2} [E} - \rho^{\mathbf{1/2}}\mathbf{O]} = \mathbf{U^T F_{n/2} G} + \mathbf{L^T F_{n/2} H.} \qquad (10)$$

By definition $\rho^{1/2} = \mathbf{F}^*_{n/2}\, \mu^{-1/2}_{n/2}\, \mathbf{F}_{n/2}$ is one of the square roots of ρ (2^n in number). It could be presented in the following way:

$$\rho^{1/2} = \sum_{k=0}^{n/2-1} e^{-j\frac{2\pi k}{n}}\, \vec{\vec{\mathbf{f}}}_k\, \vec{\vec{\mathbf{f}}}_k^* = \sum_{k=0}^{n/2-1} e^{-j\frac{2\pi k}{n}}\, \mu^k\, \frac{2}{n}\vec{\mathbf{1}}\vec{\mathbf{1}}^T\, \mu^{-k} = \sum_{k=0}^{n/2-1} e^{-j\frac{2\pi k}{n}}\, \mu^k\, \frac{2}{n}\Gamma_0(\rho)\mu^{-k}$$

It follows from the property of the commutator of a Heisenberg-Weyl's group [3] that $\mu^k \rho^l = \rho^l \mu^k e^{\frac{2\pi}{n/2} k\, l}$ (the dimension is n/2), so that finally:

$$\rho^{1/2} = \sum_{k=0}^{n/2-1} r_k \rho_{n/2}^{k}; \quad r_k = \frac{2}{n}\,(1 + j\,\mathrm{ctg}\,(2k-1)\frac{\pi}{n}). \tag{11}$$

From (11) it follows for the two operators **G** and **H** that:

$$\mathbf{G} = (\mathbf{E} + \rho^{1/2}\mathbf{O}) = \mathbf{E}(1+\beta); \quad \mathbf{H} = (\mathbf{E} - \rho^{1/2}\mathbf{O}) = \mathbf{E}(1-\beta); \quad \beta = \sum_{k=0}^{n/2-1} r_k \rho_n^{2k-1}. \tag{12}$$

If (11) is taken into consideration, it is easy to show that the following relations are valid [6]:

$$\mathbf{G}\mathbf{H}^* = 0; \quad \mathbf{G}\mathbf{G}^* = \mathbf{H}\mathbf{H}^* = \mathbf{2}; \Rightarrow \mathbf{G}^+ = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{G}^*; \quad \mathbf{H}^+ = \frac{1}{2}\mathbf{H}^*;$$
$$\beta = \mathbf{E}^{\mathbf{T}}\rho^{1/2}\mathbf{O} + \mathbf{O}^{\mathrm{T}}\rho^{1/2*}\mathbf{E}; \tag{13}$$

The two projectors onto $\mathrm{ran}\,(\mathbf{G}^*)$ and $\mathrm{ran}\,(\mathbf{H}^*)$ (the pseudoinverse [4] matrices of **G**, **H** are $\mathbf{G}^+$ and $\mathbf{H}^+$):

$$\mathbf{G}^+\mathbf{G} = \frac{1}{2}(1+\beta); \quad \mathbf{H}^+\mathbf{H} = \frac{1}{2}(1-\beta) \tag{14}$$

These projectors are orthogonal resolution of the identity obviously. It can be obtained from (11) and (12) that:

$$\beta = \rho\,(1+\kappa^2) + j\,\kappa; \quad \beta^2 = 1; \quad \beta^* = \beta. \tag{15}$$

The automorphism β is an involution (its square is identity) and it is a Hermitian morphism i.e. coincides with its Hermitian-conjugated. In that case for the orthogonal projectors (1+β)/2 and (1-β)/2 we will have that:

$$\mathbf{F}\,(\frac{1+\beta}{2}) = \mathrm{diag}\;(\vec{\mathbf{1}}, \vec{\mathbf{0}})\,\mathbf{F},$$
$$\mathbf{F}\,(\frac{1-\beta}{2}) = \mathrm{diag}\;(\vec{\mathbf{0}}, \vec{\mathbf{1}})\,\mathbf{F}. \tag{16}$$

The first projector “cuts off” the upper n/2 co-ordinates of the spectrum of the signal, and the second one – the lower n/2 co-ordinates (Fourier transforms are “one sided”). This permits in (10) and (12) to decimate the signals because they have halfband spectrum in comparison with the initial signal. This “spectral interpretation” of the FFT is very interesting and is nearly obvious. Of course, the construction of the filters and especially applying the Hilbert Transform for these filters is not so easy to be “invented” from some general notions.

Formulae (15) are an interesting connection of the Fourier Transform and Hilbert Transform. They give us possibilities to construct new different variants of the FFT with better properties than the old ones.

CONCLUSIONS

This paper deals with the connections between the Fourier and Hilbert transforms. On the base of the theory of groups approach new form of the analyzing filters of the Fast Fourier

Transform was obtained. This broadens the boundaries for construction of new variants of the FFT with smaller computing complexity. The same is true for the Hilbert transform too.

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