Invariant Spaces and Cosine Transforms DCT- 2 and DCT- 6

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Abstract: *In paper two types of the discrete cosine (and sine) transforms (DCT/DST) are analyzed. These transforms are useful for many applications. It is shown that if an operator, connected with the Discrete Fourier Transform (DFT), is referred to an appropriate basis it takes block-diagonal form. These blocks coincide with DCT-2/DST-2 for even dimensions of the signals' space and with DCT-6/DST-6 for odd ones. These results complete investigating the full structure of the DCT/DST. The other transforms DCT-3/DST-3 and DCT-7/DST-7 have simple connection with these one – they are transposed to them.*

Key words: Cosine transforms, orthogonality, signal processing, fast transforms, filter banks, characters of groups and theory of groups.

INTRODUCTION

The discrete cosine transform (DCT) uses n real basis vectors $\{\vec{\mathbf{c}}_m\}$ with cosine coordinates. These basis vectors are orthogonal. For example the k-component of $\vec{\mathbf{c}}_m$ in DCT-2 is $\frac{2}{\sqrt{n}}\cos(\frac{2\pi}{n}k.(m+\frac{1}{2}))$. There are eight different types of DCT.

Ahmed, Natarajan, and Rao found the first cosine transform in 1974. This is the so-called DCT-2 [1][3][12]. There are four basic types - from DCT-1 to DCT-4. This basic set was expanded in 1985 with new four transforms – from DCT-IO to DCT-IVO by Wang and Hunt [2][3].

All DCT are orthogonal transforms and a usual proof is the direct calculation of inner products of their basis vectors, applying trigonometric identities [3]. The proof of orthogonality is obtained in the Strang's paper [3] by second indirect but neat way. The basis vectors are actually eigenvectors of symmetric second-difference matrices by different boundary conditions. Orthogonality is proofed automatically (matrices are symmetric) and all DCTs are connected in fixed structure.

Does more direct way exist to obtain these transforms, connecting them in joint structure, proofing orthogonality and giving fast realizations? The first answer of this question is given in [4][5].

The objective of this paper is to get the final answer of this question, describing the connections of DCT-2 with DCT-6. The last transforms DCT-3 and DCT-7 are their transposed variant.

CONVOLUTION AND THE DIHEDRAL GROUP

The input and output signals of a linear time-invariant system are connected by the convolution operation [6]:

$$\mathbf{y} = \mathbf{x} * \mathbf{h} \quad . \qquad (1)$$

Here **h** is the impulse response of the system. The sets of the real numbers **R,** integer numbers **Z** and the integer numbers – multiple of some integer number **n** (i.e. **nZ),** with addition as a binary operation, are groups [5]. The signals are functions usually defined on the **R**, **Z**, the torus group **T** = **R**/**Z**, the residue system (mod n) **Z**/**nZ**, or their Cartesian products. As such they are elements of some functional space, most often a Hilbert space **H**, which is supplied with the form **(x|y)** that takes values in the field of the complex numbers C. This form, called an inner or scalar product, is Hermitian and positive definite [9].

If $\mathbf{L}^2$(a, b) denotes the space of square summable functions on the interval (a, b), and $\mathbf{L}^2(\mathbf{Z}/n\mathbf{Z})$ denotes n-dimensional complex vector space of functions (vectors), that we have a Hilbert spaces with inner products [6][9][10]

$$(x\,|\,y) = \int_a^b x^*(t)\,y(t)\,dt, \qquad (\vec{\mathbf{x}}|\,\vec{\mathbf{y}}) = \sum_{k\in Z/nZ} \mathbf{x}_k^* . \mathbf{y}_k \ . \tag{2}$$

The convolution (1) could be written as an inner product if the right shift operator ρ and the sign operator σ are used: $\rho : x(t) \to x(t-1), \quad \sigma : x(t) \to x(-t)$. If x, h ∈ $\mathbf{L}^2(-\infty,\infty)$, then

$$(x\,|\,\rho^t\ \sigma h) = \int_{-\infty}^{\infty} x(t)\,h(t-\tau)\,d\tau = x * h.$$

In the canonical basis of $\mathbf{L}^2(\mathbf{Z}/n\mathbf{Z})$, formed by vectors, $\{\vec{\mathbf{e}}_k\}$, $\vec{\mathbf{e}}_k = \left[\delta_{l,\,k}\right]$, l, k = 0,1,...,(n-1) (mod n), ($\delta_{l,\,k}$ is the Kronecker's symbol), the two endomorphisms have the following (orthogonal) matrices [11]:

$$\rho_n = \lfloor \delta_{k-1,l} \rfloor,\ \sigma_n = \lfloor \delta_{k,n-l} \rfloor_{k,l=0,1,\ldots,n-1(\bmod n)},$$

$$\rho_4 = \begin{bmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{bmatrix}, \quad \sigma_4 = \begin{bmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0 \end{bmatrix}. \tag{3}$$

In this case the k-th coordinate of the output vector $\vec{\mathbf{y}}$ in (1) has the form

$$y_k = (\vec{\mathbf{x}}\,|\,\rho^k \sigma\,\vec{\mathbf{h}}), \tag{4}$$

and the convolution is cyclic. In (4) ρ and σ define a linear representation of the dihedral group $\mathbf{D}_n$ [7][9]:

$$D_n = \langle \sigma, \rho \,|\, \sigma^2 = \rho^n = (\sigma\rho)^2 = 1 \rangle. \tag{5}$$

One verifies from (3) that the matrices of two endomorphisms satisfy the defining relations of $\mathbf{D}_n$. The group $\mathbf{D}_n$ has a second presentation, which is isomorphic to (5) $(\sigma \to \sigma, \quad \hat{\sigma} \to \sigma\rho)$:

$$D_n = \langle \sigma, \hat{\sigma} \,|\, \sigma^2 = \hat{\sigma}^2 = (\sigma\hat{\sigma})^n = 1 \rangle. \tag{5b}$$

In this case $\hat{\sigma}_n = \sigma_n \rho_n$ and if n = 4

$$\hat{\sigma}_4 = \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix}.$$

It follows from (5) that not only the obvious symmetry σ (which reflects functions in ordinate axis) is involution i.e. $\sigma^2 = 1$. Involutions are the elements of dihedral group $\rho^\kappa\sigma$ in (4), that reflects in the vertical lines $t = \frac{k}{2}$. These "parallel lines" coincide in discrete case with diameters of the unit circle. If we apply the Strang's terminology [6], for even k this symmetry realizes “meshpoint (or whole-sample) symmetry”, and for odd k - “midpoint (or half-sample) symmetry”. The most simple representatives of these two classes are σ (reflects in the ordinate) and σ.ρ ~ $\hat{\sigma}$ (reflects in the vertical line t = - ½) i.e. the two generators of the dihedral group into the second presentation of this group (5b).

DISCRETE COSINE/SINE TRANSFORM - (DCT- 2/DST- 2) AND (DCT- 6/DST- 6)

It's easy to be shown that σ and the identity 1 have two presentations:

$$\begin{aligned}\sigma &= \sum_{0\le k<n} \rho^k \vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^T \rho^k = \sum_{0\le k<n} \rho^{-k} \vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^T \rho^{-k}, \\ 1 &= \sum_{0\le k<n} \rho^k \vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^T \rho^{-k} = \sum_{0\le k<n} \rho^{-k} \vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^T \rho^{k}\ .\end{aligned} \tag{6}$$

Here $\vec{\boldsymbol{\delta}} = [\,1,0,0,...,0]^T$ is the n-dimensional vector of Dirac. Let's introduce two operators - μ and **A** (here **F** = **C** – j **S** is Discrete Fourier Transform's Operator – DFT) [3][9]:

$$\mu_n = \mathrm{diag}(1, e^{j\frac{2\pi}{n}}, ..., e^{j\frac{2\pi}{n}k}, ..., e^{j\frac{2\pi}{n}(n-1)}),\ \rho^k\,\mathbf{F} = \mathbf{F}\,\mu^k, \quad \mathbf{F}\,\rho^k = \mu^{-k}\,\mathbf{F}. \tag{7}$$

$$\mathbf{A} = \mu^{-1/2}\ \mathbf{F}\,; \quad w = e^{-j\,2\pi/n}$$

Obviously **A** is a unitary operator [8], i.e. $\mathbf{A}\,\mathbf{A}^* = \mathbf{A}^*\,\mathbf{A} = 1$ and

$$\mathbf{A} = 1/\sqrt{n}\ \sum_{0\le k,l<n} e^{-j\,2\pi\,k\,(l+1/2)/n} \rho^{\,k} \vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^T \rho^{-l} \tag{8}$$

$$\mathbf{A}\,\mathbf{A}^{\mathbf{T}} = \mu^{-1/2}\ \mathbf{F}\ \mathbf{F}\,\mu^{-1/2} = \mu^{-1/2}\ \sigma\ \mu^{-1/2} = 2\,\vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^T - \sigma\,; \mathbf{A}^{\mathbf{T}}\mathbf{A} = \mathbf{F}\mu^{-1}\,\mathbf{F} = \mathbf{F}\ \mathbf{F}\rho = \sigma\rho = \hat{\sigma}$$

$$\mathbf{A}\,\mathbf{A}^{\mathbf{T}}(\mathbf{A})\mathbf{A}^{\mathbf{T}}\mathbf{A} = \mathbf{A}(\mathbf{A}^{\mathbf{T}}\mathbf{A})\mathbf{A}^{\mathbf{T}}\mathbf{A} = \mathbf{A}(\mathbf{A}^{\mathbf{T}}\mathbf{A})^{+}\,\mathbf{A}^{\mathbf{T}}\mathbf{A} = \mathbf{A}\,\mathbf{A}^{+}\mathbf{A} = \mathbf{A}$$

Here $\mathbf{H}^{+}$ is pseudoinverse matrix of **H** [10]. It's easy to be obtained, that

$$\mu^{-1/2}\ \sigma\ \mu^{-1/2} = \vec{\boldsymbol{\delta}}\,\vec{\boldsymbol{\delta}}^T - \sum_{0<k<n} \rho^k\,\vec{\boldsymbol{\delta}}\,\vec{\boldsymbol{\delta}}^T \rho^k = 2\,\vec{\boldsymbol{\delta}}\,\vec{\boldsymbol{\delta}}^T - \sigma = \overline{\sigma};$$

$$\mathbf{A} = \overline{\sigma}\ \mathbf{A}\ \hat{\sigma}. \tag{9}$$

From here one can get some important conclusions. As $\overline{\sigma}$ and $\hat{\sigma}$ are involutions (i.e. $\overline{\sigma}^2 = 1$, $\hat{\sigma}^2 = 1$) the two projectors onto the invariant subspaces [7][9] of these operators are respectively:

$$p1 = \frac{1+\overline{\sigma}}{2}; \quad q1 = \frac{1-\overline{\sigma}}{2}; \quad p2 = \frac{1+\hat{\sigma}}{2}; \quad q2 = \frac{1-\hat{\sigma}}{2}; \tag{10}$$

Dimensions of the spaces onto which they are projecting coincide with the traces of these projectors and are respectively (n/2, n/2), if n is even and ((n+1)/2, (n-1)/2), when n is odd.

These two projectors have one more property – the first (n/2, n/2) columns - if n is even, and the first ((n+1)/2, (n-1)/2) - if n is odd, are orthogonal. About p and q if n = 4 we have respectively:

$$\begin{pmatrix}1&0&0&0\\0&\frac{1}{2}&0&-\frac{1}{2}\\0&0&0&0\\0&-\frac{1}{2}&0&\frac{1}{2}\end{pmatrix},\begin{pmatrix}0&0&0&0\\0&\frac{1}{2}&0&\frac{1}{2}\\0&0&1&0\\0&\frac{1}{2}&0&\frac{1}{2}\end{pmatrix},\begin{pmatrix}\frac{1}{2}&0&0&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{2}&0\\0&\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&0&0&\frac{1}{2}\end{pmatrix},\begin{pmatrix}\frac{1}{2}&0&0&-\frac{1}{2}\\0&\frac{1}{2}&-\frac{1}{2}&0\\0&-\frac{1}{2}&\frac{1}{2}&0\\-\frac{1}{2}&0&0&\frac{1}{2}\end{pmatrix}$$

If one constructs a bases from these orthogonal vectors, the projectors will get diagonal form (with n/2 x n/2 ones-zeros and zeros-ones on the diagonal for the even case and (n+1)/2 x (n-1)/2 – for the odd case; $\bar{\sigma}$ and $\hat{\sigma}$ will get diagonal form with –1 on the place of the zeros of the projectors and **A** from (9) – block-diagonal form.

The endomorphism of the transition to this basis β for n – even will get the form (after normalizing the basis vectors):

$$p1=\vec{\boldsymbol{\delta}}\,\vec{\boldsymbol{\delta}}^{T}+(1/2)\sum_{0\le k<n}((\rho^{k}-\rho^{-k})\,\vec{\boldsymbol{\delta}}\,\vec{\boldsymbol{\delta}}^{T}\rho^{-k};\; q1=(1/2)\sum_{0<k<n}((\rho^{k}+\rho^{-k})\,\vec{\boldsymbol{\delta}}\,\vec{\boldsymbol{\delta}}^{T}\rho^{-k};$$

$$\beta=1/\sqrt{2}\sum_{0\le k<n/2}(\rho^{k}\vec{\boldsymbol{\delta}}\,\vec{\boldsymbol{\delta}}^{T}\rho^{-k}(1+\rho^{-n/2})+\rho^{-k-1}\vec{\boldsymbol{\delta}}\,\vec{\boldsymbol{\delta}}^{T}\rho^{-k}(1-\rho^{-n/2})\qquad(11)$$

For the next projectors can be written:

$$p2=(1/2)\sum_{0\le k<n}((\rho^{k}+\rho^{-(k+1)})\vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^{T}\rho^{-k};\quad q2=(1/2)\sum_{0\le k<n}((\rho^{k}-\rho^{-(k+1)})\vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^{T}\rho^{-k};$$

$$\alpha=\vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^{T}+\rho^{n/2}\vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^{T}\rho^{-(n-1)}+\frac{1}{\sqrt{2}}\sum_{0<k<n/2}(\rho^{k}\vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^{T}\rho^{-k}(1+\rho^{-(n/2-1)})-\rho^{-k}\vec{\boldsymbol{\delta}}\vec{\boldsymbol{\delta}}^{T}\rho^{-k}(1-\rho^{-(n/2-1)})\qquad(11b)$$

From (9) and (11) we have (α and β are orthogonal):

$$\alpha^{T}\mathbf{A}\beta=\alpha^{T}\bar{\sigma}\mathbf{A}\hat{\sigma}\beta=\alpha^{T}\bar{\sigma}\,\alpha\alpha^{T}\mathbf{A}\beta\beta^{T}\hat{\sigma}\beta=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix}=\begin{bmatrix}A_{11}&-A_{12}\\-A_{21}&A_{22}\end{bmatrix};\qquad(12)$$

$$\alpha^{\mathbf{T}}\bar{\sigma}\;\alpha=diag(1,1,\ldots 1,-1,-1,\ldots-1);\;\beta^{T}\hat{\sigma}\,\beta=diag(1,1,\ldots 1,-1,-1,\ldots-1).$$

From (12) it's follows that the two anti-diagonal blocks (with respective dimensions) are zero:

$$A_{12}=0;\quad A_{21}=0;$$

A takes the block-diagonal form indeed. More over from the fact that **A** is unitary matrix follows that A_{11} is a real matrix and A_{22} is clear imaginary one. For n = 4 $\alpha^{T}\bar{\sigma}\alpha$ *and* $\beta^{T}\hat{\sigma}\beta$ have the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It could be obtained that for even n:

$$\mathbf{A^T}\ \alpha = [\vec{\mathbf{c}}_\mathbf{0}/\sqrt{2} \mid \vec{\mathbf{c}}_\mathbf{1} \mid \ldots \mid \vec{\mathbf{c}}_\mathbf{n/2-1} \mid -j\vec{\mathbf{s}}_1 \mid -j\vec{\mathbf{s}}_2 \mid \ldots \mid -j\vec{\mathbf{s}}_\mathbf{n/2}\ /\sqrt{2}] \tag{13}$$

$$\vec{\mathbf{c}}_\mathbf{l} = \sqrt{\frac{2}{n}}\left[\cos[\frac{2\pi}{n}(k+1/2)l]\right]; \quad \vec{\mathbf{s}}_\mathbf{l} = \sqrt{\frac{2}{n}}\left[\sin[\frac{2\pi}{n}(k+1/2)l]\right]$$

$$\kappa = 0, 1, \ldots, n-1; \quad l = 0, 1, \ldots, n/2$$

Eventually

$$\alpha^\mathbf{T}\mathbf{A}\ \beta = \begin{bmatrix} \mathbf{C}2 & 0 \\ 0 & -j\ \mathbf{S}2 \end{bmatrix} \tag{14}$$

$$\mathbf{C}2 = \mathbf{DC2}.\left[\frac{2}{\sqrt{n}}\cos[\frac{2\pi}{n}k\,(l+1/2)\right]; \quad \mathbf{S}2 = \mathbf{DS2}.\left[\frac{2}{\sqrt{n}}\sin[\frac{2\pi}{n}k\,(l+1/2)\right]$$

$$0 \le k, l < n/2; \qquad 0 < k \le n/2;\ 0 \le l \le n/2-1$$

$$\mathbf{DC2} = diag(1/\sqrt{2}, 1, 1, \ldots, 1); \qquad \mathbf{DS2} = diag(1, 1, 1, \ldots, 1, 1/\sqrt{2})$$

We received simultaneously two transforms - DCT-2 and DST-2. The same form can be obtained for n – odd i.e. for DCT-6 and DST-6. The two another couples DCT-3 / DST-3 and DCT-7/DST-7 can be received with transposing of these one.

CONCLUSIONS

In this paper two types of cosine/sine transforms DCT-2/DST-2 and DCT-6/DST-6 was analyzed. On the base of the theory of groups approach [9] and the invariant spaces of the operator **A** its block-diagonal form was obtained. The two blocks of **A** coincide with the well-known cosine/sine transforms. This result demonstrates the common genesis of all cosine/sine transforms and allows fast realizations to be received. On the base of proposed approach was accomplished (if we include our works [4][5]) the full analysis of all types of cosine/sine transforms.

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