A Computer Implementation of the Generalized Partial Credit Model

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Abstract: *Generalised Partial Credit Model (**GPCM**) is used in the case of a test with graded response categories. It is a generalization of both – the graded response model and the two-parametric logistic model and in this way it seems to be one of the most useful models with respect to the psychological scales. Here we present a **computer implementation of the GPCM** as well as we give a brief description of the model itself and of the way it is implemented.*
Key words: *Generalized Partial Credit Model, MMLE/EM.*

INTRODUCTION

Generalized Partial Credit Model has been introduced by ***EiJi Muraki*** (1992). **GPCM** is based on the well-known partial credit model with additional assumption for different discrimination rate of items. Suppose we are given a test with n items and the i-th item has m_i categories $h=1,2,\ldots,m_i$. Denote as usual by $P_{ih}(\theta)$ the probability of a person with ability θ to answer by the h-th category of the i-th item. The **GPCM** assumes that

$$\frac{P_{ih}(\theta)}{P_{i,h-1}(\theta)+P_{ih}(\theta)}=\frac{\exp\left[a_i(\theta-b_{ih})\right]}{\exp\left[a_i(\theta-b_{ih})\right]+1}, \qquad (1)$$

which means that the choice between two subsecuent categories $h-1$ and h conforms to the two-parametric logistic model.

Here we will describe the original approach of Muraki (see [3]). The parameter b_{ih} will be decomposed as follows $b_{ih}=b_i-d_{ih}$ where b_i stands for the difficulty parameter of the i-th item and d_{ih} are the category threshlold parameters. We choose to set $d_{i1}=0$ to remove the indefiniteness in this decomposition. Denote

$$z_{ih}(\theta)=a_i(\theta-b_i+d_{ih}), \qquad (2)$$

$$z^{+}_{ih}(\theta)=\sum_{\nu=1}^{h} z_{i\nu}(\theta)=a_i\left[h(\theta-b_i)+d_{i1}+d_{i2}+\cdots+d_{ih}\right]. \qquad (3)$$

Then by (1) for the probabilities $P_{ih}(\theta)$ one can find that

$$P_{ih}(\theta)=\frac{\exp\left[z^{+}_{ih}(\theta)\right]}{\sum_{\nu=1}^{m_i}\exp\left[z^{+}_{i\nu}(\theta)\right]}. \qquad (4)$$

Given the test results the problem is to estimate the model parameters afterwards the calibrated model can be used in the practice.

PARAMETER ESTIMATION

Let S be the number of the examinees. Let also u_{ihl} be the anser indicator: for the i-th item $u_{ihl}=1$ when l-th person responses by category h and $u_{ihl}=0$ otherwise. Then the natural likelyhood function for the l-th response pattern is

$$LH_l(\theta)=\prod_{i=1}^{n}\prod_{h=1}^{m_i}\left[P_{ih}(\theta)\right]^{u_{ihl}}. \qquad (5)$$

We will use the Marginal Maximum Likelihood Estimation (**MMLE**) principal hereafter in which it is assumed that any person has an ability distribution – usually normal standard with a density function $\varphi(\theta)$. Note that the same ability distribution is assumed for all examinees. The ***marginal*** log-likelyhood function for the l-th response pattern is

$$MLH_l = \ln\left[\int_{-\infty}^{\infty} LH_l(\theta)\varphi(\theta)d\theta\right] = \ln\left[\int_{-\infty}^{\infty}\prod_{i=1}^{n}\prod_{h=1}^{m_i}\left[P_{ih}(\theta)\right]^{u_{ihl}}\varphi(\theta)d\theta\right], \quad (6)$$

and the overall log-likelyhood function is

$$MLH = \sum_{l=1}^{S}\ln\left[\int_{-\infty}^{\infty} LH_l(\theta)\varphi(\theta)d\theta\right]. \quad (7)$$

At this point we have to estimate only the item parameters. Let $\partial_{i\xi}$ denotes a differentiation with respect to some parameter of the i-th item. Then

$$\partial_{i\xi}MLH = \sum_{l=1}^{S}\frac{\int_{-\infty}^{\infty}\partial_{i\xi}LH_l(\theta)\varphi(\theta)d\theta}{\int_{-\infty}^{\infty}LH_l(\theta)\varphi(\theta)d\theta}. \quad (8)$$

On the other hand

$$\partial_{i\xi}LH_l(\theta) = LH_l(\theta)\sum_{h=1}^{m_i}\frac{\partial_{i\xi}P_{ih}(\theta)}{P_{ih}(\theta)}u_{ihl}, \quad (9)$$

therefore

$$\partial_{i\xi}MLH = \sum_{l=1}^{S}\frac{\int_{-\infty}^{\infty}LH_l(\theta)\sum_{h=1}^{m_i}\frac{\partial_{i\xi}P_{ih}(\theta)}{P_{ih}(\theta)}u_{ihl}\varphi(\theta)d\theta}{\int_{-\infty}^{\infty}LH_l(\theta)\varphi(\theta)d\theta}. \quad (10)$$

We will use a gaussian numerical integration, with $F = 21$ nodes in this case, with weights w_f and nodes x_f, $f = 1,2,\ldots,F$. These values can be found in [5]. Then

$$\int_{-\infty}^{\infty}LH_l(\theta)\varphi(\theta)d\theta \approx \sum_{f=1}^{F}LH_l(x_f)w_f = \mathbf{P}_l. \quad (11)$$

Set also

$$\mathbf{r}_{ihf} = \sum_{l=1}^{S}\frac{LH_l(x_f)w_f u_{ihl}}{\mathbf{P}_l}, \ \mathbf{N}_f = \sum_{l=1}^{S}\frac{LH_l(x_f)w_f}{\mathbf{P}_l}. \quad (12)$$

Now (10) turns into the following numerical form

$$\partial_{i\xi}MLH = \sum_{f=1}^{F}\sum_{h=1}^{m_i}\mathbf{r}_{ihf}\frac{\partial_{i\xi}P_{ih}(\theta)}{P_{ih}(\theta)}. \quad (13)$$

Maximum likelihood estimation for the item parameters is obtained by solving of all the equations $\partial_{i\xi}MLH = 0$, i.e.

$$\sum_{f=1}^{F}\sum_{h=1}^{m_i}\mathbf{r}_{ihf}\frac{\partial_{i\xi}P_{ih}(\theta)}{P_{ih}(\theta)} = 0, \quad (14)$$

for all $i = 1,2,\ldots,n$, $\xi = a_i, b_i, d_{i1},\ldots,d_{im_i}$. For example, about a_i parameter we get the following equations (see [3])

$$\mathbf{g}_{a_i} = \frac{1}{a_i}\sum_{f=1}^{F}\sum_{h=1}^{m_i}\mathbf{r}_{ihf}\left[z_{ih}^{+}\left(x_f\right) - \overline{z}_i^{+}\left(x_f\right)\right] = 0, \tag{15}$$

where

$$\overline{z}_i^{+}\left(\theta\right) = \sum_{h=1}^{m_i} z_{ih}^{+}\left(\theta\right)P_{ih}\left(\theta\right). \tag{16}$$

In the same way about b_i parameters we get the equations

$$\mathbf{g}_{b_i} = a_i\sum_{f=1}^{F}\sum_{h=1}^{m_i}\mathbf{r}_{ihf}\left[-h + T_i\left(x_f\right)\right] = 0, \tag{17}$$

where

$$T_i\left(\theta\right) = \sum_{h=1}^{m_i} hP_{ih}\left(\theta\right). \tag{18}$$

About d_{ih} parameters we get the equations

$$\mathbf{g}_{d_{ih}} = a_i\sum_{f=1}^{F}\sum_{k=h}^{m_i}\left[\mathbf{r}_{ikf} - P_{ik}\left(x_f\right)\sum_{\nu=1}^{m_i}\mathbf{r}_{i\nu f}\right] = 0. \tag{19}$$

We solve **MMLE** equations (15), (17) and (19) by a two-stage **EM** (expectation maximization) scheme. Start from initial zero values and do the next steps.

1. E-step: recalculate $\mathbf{r}_{ihl}$ (and $\mathbf{N}_f$).

2. M-step: Resolve (15)-(17)-(19) for all items. Here firstly two equations (15) and (17) are solved afterwards $m_i - 1$ equations (19) are solved.

3. Repeat steps 1 and 2 until some stop criterion is met.

In the **M**-step we use the Newton-Raphson algorithm for which we need to know the Hessian matrix $\mathbf{H}$ for the i-th item. One can find for a_i and b_i that (see [3])

$$\mathbf{H}_{a_i a_i} = -\frac{1}{a_i^2}\sum_{f=1}^{F}\mathbf{N}_f\sum_{h=1}^{m_i}P_{ih}\left(x_f\right)\left[z_{ih}^{+}\left(x_f\right) - \overline{z}_i^{+}\left(x_f\right)\right]^2, \tag{20}$$

$$\mathbf{H}_{b_i b_i} = -a_i^2\sum_{f=1}^{F}\mathbf{N}_f\sum_{h=1}^{m_i}P_{ih}\left(x_f\right)\left[-h + T_i\left(x_f\right)\right]^2, \tag{21}$$

$$\mathbf{H}_{a_i b_i} = -\sum_{f=1}^{F}\mathbf{N}_f\sum_{h=1}^{m_i}P_{ih}\left(x_f\right)\left[z_{ih}^{+}\left(x_f\right) - \overline{z}_i^{+}\left(x_f\right)\right]\left[-h + T_i\left(x_f\right)\right]. \tag{22}$$

As well, for $h' \le h$, we get

$$\mathbf{H}_{d_h d_{h'}} = -a_i^2\sum_{f=1}^{F}\mathbf{N}_f\left[\sum_{k=h}^{m_i}P_{ik}\left(x_f\right)\right]\left[1 - \sum_{k=h'}^{m_i}P_{ik}\left(x_f\right)\right]. \tag{23}$$

Remember that, in the Newton-Raphson method, the succsesive steps are done according to the formula $\boldsymbol{\xi}_{t+1} = \boldsymbol{\xi}_t - \mathbf{H}_t^{-1}\mathbf{g}_t$.

Having once the values of the item parameters we can give an estimation $\hat{\theta}_l$ of the ability level for the l-th examinee using **EAP** (expected a posteriori) technique

$$\hat{\theta}_l = \frac{\sum_{f=1}^{F} x_f LH_l\left(x_f\right) w_f}{\sum_{f=1}^{F} LH_l\left(x_f\right) w_f}, \quad l = 1,2,\ldots,S. \tag{24}$$

MODIFICATIONS

Presented **GPCM** model admits straightforward modifications. First if we set $a_i = 1$ for all items we obviously receive the partial credit model. Another modification is obtained when one requires treshholds d_{ih} to be placed in equal intervals, i.e. $d_{ih} = d_{i1} + \left(i-1\right)h_i$. Furthermore one can require all the treshholds to be placed in equal intervals, i.e. $d_{ih} = d_{i1} + \left(i-1\right)h$ with the same h for all items. These modifications are included in our computer implementation and can be used in place of the original model.

EXAMPLE

An illustrative example is considered here. Data was created by the ***data generator***. The example contains **10** items with **5** categories and **100** examinees. Here follow the results after **30** repeats of the **EM** cycle.

Table 1.

item	a	b	d2	d3	d4	d5	eqval	grad
1	1.225	0.716	0.928	0.122	-1.119	-1.145	0.00000006	0.00000532
2	1.264	0.175	1.417	0.282	-0.473	-1.537	0.00000006	0.00000606
3	0.482	0.854	1.080	-0.871	-0.461	-1.672	0.00000005	0.00000261
4	1.144	0.189	0.718	0.886	-0.277	-1.691	0.00000005	0.00000571
5	1.130	0.561	1.149	-0.214	-0.475	-1.541	0.00000005	0.00000510
6	0.904	0.451	1.124	0.326	-0.591	-1.981	0.00000006	0.00000407
7	1.546	0.194	1.064	0.234	-0.491	-1.021	0.00000005	0.00000846
8	1.032	0.557	0.865	0.399	-0.735	-1.358	0.00000006	0.00000511
9	1.473	0.226	1.526	-0.255	-0.783	-1.080	0.00000005	0.00000673
10	0.863	0.496	1.289	0.163	-0.845	-2.075	0.00000005	0.00000353

Column **eqval** shows the sum of the absolute value of all the left-hand sides of the equations solved for the corresponding item. In the same way column **grad** shows the value of the gradients. Normally $d_{i2}, d_{i3}, \ldots, d_{im_i}$ forms a decreasing sequence but this is not obligatory for the model validity.

FIT ANALYSIS

The goodness of model fit can be estimated by calculatig of a proper chi-square indexes. Define several intervals $\Delta_1, \Delta_2, \ldots, \Delta_W$, which cover the θ interval. Let ρ_{wih} is the number of examinees with h-th category response on item i, whose **EAP** ability $\hat{\theta}$ belongs to the w-th interwal, $w = 1,2,\ldots,W$, and $N_{wi} = \sum \rho_{wih}$. Let also $\overline{\theta}_w$ be the interval mean. Then the value

$$2\sum_{w=1}^{W}\sum_{h=1}^{m_i} \rho_{wih} \ln\left[\frac{\rho_{wih}}{N_{wi} P_{ih}\left(\overline{\theta}_w\right)}\right] \tag{25}$$

presents a chi-square distributed variable with $W\left(m_i - 1\right)$ degrees of freedom. The null-hypothesis is that the corresoponding item is not consistent with respect to the group of the other items. Rejecting the null-hypothesis, based on the value of (25), we state that the i-th item is consistent with the scale.

CONCLUSIONS AND FUTURE WORK

Obvioysly the **GPCM** model is of a major importance. Fortunately the algorithm presented above works well. It is interesting to experiment with some quasi-Newton method instead of the Newton-Raphson method which is the next purpose of the authors on this topic.

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