A Computer Implementation of the Generalized Partial Credit Model

Lyubomir Christov, Dimiter Tsvetkov

Abstract: Generalised Partial Credit Model (GPCM) is used in the case of a test with graded response categories. It is a generalization of both – the graded response model and the two-parametric logistic model and in this way it seems to be one of the most useful models with respect to the psychological scales. Here we present a computer implementation of the GPCM as well as we give a brief description of the model itself and of the way it is implemented.

Key words: Generalized Partial Credit Model, MMLE/EM.

INTRODUCTION

Generalized Partial Credit Model has been introduced by EiJi Muraki (1992). GPCM is based on the well-known partial credit model with additional assumption for different discrimination rate of items. Suppose we are given a test with $n$ items and the $i$-th item has $m_i$ categories $h = 1, 2, \ldots, m_i$. Denote as usual by $P_{ih}(\theta)$ the probability of a person with ability $\theta$ to answer by the $h$-th category of the $i$-th item. The GPCM assumes that

$$\frac{P_{ih}(\theta)}{P_{i,h-1}(\theta) + P_{ih}(\theta)} = \exp\left[ a_i(\theta - b_{ih}) \right] + 1,$$

which means that the choice between two subsequent categories $h-1$ and $h$ conforms to the two-parametric logistic model.

Here we will describe the original approach of Muraki (see [3]). The parameter $b_{ih}$ will be decomposed as follows $b_{ih} = b_i - d_{ih}$ where $b_i$ stands for the difficulty parameter of the $i$-th item and $d_{ih}$ are the category threshold parameters. We choose to set $d_{i1} = 0$ to remove the indefiniteness in this decomposition. Denote

$$z_{ih}(\theta) = a_i(\theta - b_i + d_{ih}),$$

$$z^{+}_{ih}(\theta) = \sum_{\nu=1}^{h} z_{\nu i}(\theta) = a_i\left[h(\theta - b_i) + d_{i1} + d_{i2} + \cdots + d_{ih}\right].$$

Then by (1) for the probabilities $P_{ih}(\theta)$ one can find that

$$P_{ih}(\theta) = \frac{\exp[z^{+}_{ih}(\theta)]}{\sum_{i=1}^{m_i} \exp[z^{+}_{\nu i}(\theta)]}.$$

PARAMETER ESTIMATION

Let $S$ be the number of the examinees. Let also $u_{ihl}$ be the anser indicator: for the $i$-th item $u_{ihl} = 1$ when $l$-th person responses by category $h$ and $u_{ihl} = 0$ otherwise. Then the natural likelyhood function for the $l$-th response pattern is

$$LH_i(\theta) = \prod_{j=1}^{n} \prod_{h=1}^{m_i} [P_{ih}(\theta)]^{u_{ihl}}.$$

We will use the Marginal Maximum Likelihood Estimation (MMLE) principal hereafter in which it is assumed that any person has an ability distribution – usually normal standard
with a density function \( \varphi(\theta) \). Note that the same ability distribution is assumed for all examinees. The marginal log-likelihood function for the \( l \)-th response pattern is

\[
MLH_l = \ln \left[ \int_{-\infty}^{\infty} LH_l(\theta) \varphi(\theta) d\theta \right] = \ln \left[ \int_{-\infty}^{\infty} \prod_{i=1}^{m} \prod_{h=1}^{n} P_{ih}(\theta)^{u_{ihl}} \varphi(\theta) d\theta \right],
\]

and the overall log-likelihood function is

\[
MLH = \sum_{l=1}^{S} \ln \left[ \int_{-\infty}^{\infty} LH_l(\theta) \varphi(\theta) d\theta \right].
\]

At this point we have to estimate only the item parameters. Let \( \partial_{\xi} \) denotes a differentiation with respect to some parameter of the \( i \)-th item. Then

\[
\partial_{\xi} MLH = \sum_{l=1}^{S} \frac{\int_{-\infty}^{\infty} \partial_{\xi} LH_l(\theta) \varphi(\theta) d\theta}{\int_{-\infty}^{\infty} LH_l(\theta) \varphi(\theta) d\theta}.
\]

On the other hand

\[
\partial_{\xi} LH_l(\theta) = LH_l(\theta) \sum_{h=1}^{m} \frac{\partial_{\xi} P_{ih}(\theta)}{P_{ih}(\theta)} u_{ihl},
\]

therefore

\[
\partial_{\xi} MLH = \sum_{l=1}^{S} \frac{\int_{-\infty}^{\infty} LH_l(\theta) \sum_{h=1}^{m} \frac{\partial_{\xi} P_{ih}(\theta)}{P_{ih}(\theta)} u_{ihl} \varphi(\theta) d\theta}{\int_{-\infty}^{\infty} LH_l(\theta) \varphi(\theta) d\theta}.
\]

We will use a gaussian numerical integration, with \( F = 21 \) nodes in this case, with weights \( w_f \) and nodes \( x_f \), \( f = 1, 2, \ldots, F \). These values can be found in [5]. Then

\[
\int_{-\infty}^{\infty} LH_l(\theta) \varphi(\theta) d\theta \approx \sum_{f=1}^{F} LH_l(x_f) w_f = P_l.
\]

Set also

\[
r_{lf} = \sum_{l=1}^{S} LH_l(x_f) w_f u_{ihl}, \quad N_f = \sum_{l=1}^{S} LH_l(x_f) w_f.
\]

Now (10) turns into the following numerical form

\[
\partial_{\xi} MLH = \sum_{f=1}^{F} \sum_{h=1}^{m} r_{lf} \frac{\partial_{\xi} P_{ih}(\theta)}{P_{ih}(\theta)}.
\]

Maximum likelihood estimation for the item parameters is obtained by solving of all the equations \( \partial_{\xi} MLH = 0 \), i.e.

\[
\sum_{f=1}^{F} \sum_{h=1}^{m} r_{lf} \frac{\partial_{\xi} P_{ih}(\theta)}{P_{ih}(\theta)} = 0,
\]
for all \( i = 1, 2, \ldots, n \), \( \xi = a_i, b_i, d_{il}, \ldots, d_{im_l} \). For example, about \( a_i \) parameter we get the following equations (see [3])

\[
g_{a_i} = \frac{1}{a_i} \sum_{f=1}^{F} \sum_{h=1}^{m_i} r_{ihf} \left[ z_{ih}^+ (x_f) - \overline{z}_{i}^+ (x_f) \right] = 0, \tag{15}
\]

where

\[
\overline{z}_{i}^+ (\theta) = \sum_{h=1}^{m_i} z_{ih}^+ (\theta) P_{ih} (\theta). \tag{16}
\]

In the same way about \( b_i \) parameters we get the equations

\[
g_{b_i} = a_i \sum_{f=1}^{F} \sum_{h=1}^{m_i} r_{ihf} \left[ -h + T_i (x_f) \right] = 0, \tag{17}
\]

where

\[
T_i (\theta) = \sum_{h=1}^{m_i} h P_{ih} (\theta). \tag{18}
\]

About \( d_{ih} \) parameters we get the equations

\[
g_{d_{ih}} = a_i \sum_{f=1}^{F} \sum_{k=h}^{m_i} r_{ikf} - P_{ik} (x_f) \sum_{\nu=1}^{m_i} r_{i\nu f} = 0. \tag{19}
\]

We solve MMLE equations (15), (17) and (19) by a two-stage EM (expectation maximization) scheme. Start from initial zero values and do the next steps.

1. **E-step**: recalculate \( r_{ihl} \) (and \( N_f \)).

2. **M-step**: Resolve (15)-(17)-(19) for all items. Here firstly two equations (15) and (17) are solved afterwards \( m_i - 1 \) equations (19) are solved.

3. **Repeat** steps 1 and 2 until some stop criterion is met.

In the M-step we use the Newton-Raphson algorithm for which we need to know the Hessian matrix \( H \) for the \( i \)-th item. One can find for \( a_i \) and \( b_i \) that (see [3])

\[
H_{a_i} = -\frac{1}{a_i^2} \sum_{f=1}^{F} \sum_{h=1}^{m_i} N_f \sum_{f} P_{ih} (x_f) \left[ z_{ih}^+ (x_f) - \overline{z}_{i}^+ (x_f) \right]^2, \tag{20}
\]

\[
H_{b_i} = -a_i^2 \sum_{f=1}^{F} \sum_{h=1}^{m_i} P_{ih} (x_f) \left[ -h + T_i (x_f) \right]^2, \tag{21}
\]

\[
H_{a_i} = -\sum_{f=1}^{F} \sum_{h=1}^{m_i} P_{ih} (x_f) \left[ z_{ih}^+ (x_f) - \overline{z}_{i}^+ (x_f) \right] \left[ -h + T_i (x_f) \right]. \tag{22}
\]

As well, for \( h' \leq h \), we get

\[
H_{d_{ih}d_{ih'}} = -a_i^2 \sum_{f=1}^{F} \sum_{h=1}^{m_i} P_{ih} (x_f) \left[ 1 - \sum_{k=h'}^{m_i} P_{ik} (x_f) \right]. \tag{23}
\]

Remember that, in the Newton-Raphson method, the succesive steps are done according to the formula \( \xi_{t+1} = \xi_t - H^{-1} g_t \).

Having once the values of the item parameters we can give an estimation \( \hat{\theta}_l \) of the ability level for the \( l \)-th examinee using EAP (expected a posteriori) technique.
MODIFICATIONS

Presented GPCM model admits straightforward modifications. First if we set $a_i = 1$ for all items we obviously receive the partial credit model. Another modification is obtained when one requires thresholds $d_{ih}$ to be placed in equal intervals, i.e. $d_{ih} = d_{i1} + (i - 1)h_i$.

Furthermore one can require all the thresholds to be placed in equal intervals, i.e. $d_{ih} = d_{i1} + (i - 1)h$ with the same $h$ for all items. These modifications are included in our computer implementation and can be used in place of the original model.

EXAMPLE

An illustrative example is considered here. Data was created by the data generator. The example contains 10 items with 5 categories and 100 examinees. Here follow the results after 30 repeats of the EM cycle.

Table 1.

<table>
<thead>
<tr>
<th>item</th>
<th>a</th>
<th>b</th>
<th>d2</th>
<th>d3</th>
<th>d4</th>
<th>d5</th>
<th>eqval</th>
<th>grad</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.225</td>
<td>0.716</td>
<td>0.928</td>
<td>0.122</td>
<td>-1.119</td>
<td>-1.145</td>
<td>0.00000006</td>
<td>0.00000532</td>
</tr>
<tr>
<td>2</td>
<td>1.264</td>
<td>0.175</td>
<td>1.417</td>
<td>0.282</td>
<td>-0.473</td>
<td>-1.537</td>
<td>0.00000006</td>
<td>0.00000606</td>
</tr>
<tr>
<td>3</td>
<td>0.482</td>
<td>0.854</td>
<td>1.080</td>
<td>0.326</td>
<td>-0.591</td>
<td>-1.981</td>
<td>0.00000006</td>
<td>0.00000407</td>
</tr>
<tr>
<td>4</td>
<td>1.144</td>
<td>0.189</td>
<td>0.718</td>
<td>0.886</td>
<td>-0.277</td>
<td>-1.691</td>
<td>0.00000005</td>
<td>0.00000261</td>
</tr>
<tr>
<td>5</td>
<td>1.130</td>
<td>0.561</td>
<td>1.149</td>
<td>-0.214</td>
<td>-0.475</td>
<td>-1.541</td>
<td>0.00000005</td>
<td>0.00000510</td>
</tr>
<tr>
<td>6</td>
<td>0.904</td>
<td>0.451</td>
<td>1.124</td>
<td>0.326</td>
<td>-0.591</td>
<td>-1.981</td>
<td>0.00000005</td>
<td>0.00000407</td>
</tr>
<tr>
<td>7</td>
<td>1.546</td>
<td>0.194</td>
<td>1.064</td>
<td>0.234</td>
<td>-0.491</td>
<td>-1.021</td>
<td>0.00000005</td>
<td>0.00000846</td>
</tr>
<tr>
<td>8</td>
<td>1.032</td>
<td>0.557</td>
<td>0.865</td>
<td>0.399</td>
<td>-0.735</td>
<td>-1.358</td>
<td>0.00000006</td>
<td>0.00000511</td>
</tr>
<tr>
<td>9</td>
<td>1.473</td>
<td>0.226</td>
<td>1.526</td>
<td>-0.255</td>
<td>-0.783</td>
<td>-1.080</td>
<td>0.00000005</td>
<td>0.00000673</td>
</tr>
<tr>
<td>10</td>
<td>0.863</td>
<td>0.496</td>
<td>1.289</td>
<td>0.163</td>
<td>-0.845</td>
<td>-2.075</td>
<td>0.00000005</td>
<td>0.00000353</td>
</tr>
</tbody>
</table>

Column eqval shows the sum of the absolute value of all the left-hand sides of the equations solved for the corresponding item. In the same way column grad shows the value of the gradients. Normally $d_{i2}, d_{i3}, \ldots, d_{im_i}$ forms a decreasing sequence but this is not obligatory for the model validity.

FIT ANALYSIS

The goodness of model fit can be estimated by calculating of a proper chi-square indexes. Define several intervals $\Delta_1, \Delta_2, \ldots, \Delta_W$, which cover the $\theta$ interval. Let $\rho_{wih}$ is the number of examinees with $h$-th category response on item $i$, whose EAP ability $\hat{\theta}$ belongs to the $w$-th interval, $w = 1, 2, \ldots, W$, and $N_{wi} = \sum \rho_{wih}$. Let also $\bar{\theta}_w$ be the interval mean. Then the value

$$
2 \sum_{w=1}^{W} \sum_{h=1}^{m_w} \rho_{wih} \ln \left( \frac{\rho_{wih}}{N_{wi} P_{ih} (\bar{\theta}_w)} \right)
$$

presents a chi-square distributed variable with $W(m_i - 1)$ degrees of freedom. The null-hypothesis is that the corresponding item is not consistent with respect to the group of the other items. Rejecting the null-hypothesis, based on the value of (25), we state that the $i$-th item is consistent with the scale.
CONCLUSIONS AND FUTURE WORK

Obviously the **GPCM** model is of a major importance. Fortunately the algorithm presented above works well. It is interesting to experiment with some quasi-Newton method instead of the Newton-Raphson method which is the next purpose of the authors on this topic.

REFERENCES


ABOUT THE AUTHOR

Assoc.Prof. Dimiter Petkov Tsvetkov, PhD, Department of Mathematical and Natural Sciences, National Military University "Vassil Levski", Veliko Tarnovo, Phone: +359 62 600 210, E-mail: *dimiter99@yahoo.com*.

Asist.Prof. Lyubomir Yanakiev Christov, Department of Mathematical Analysis and Applications, University of Veliko Tarnovo "St. St. Cyril and Methodius ", Veliko Tarnovo, Phone: +359 62 474 68, E-mail: *Lyubo60@yahoo.com*.